STEADY THERMAL CONTACT OF A HALF STRIP AND A STRIP

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The steady temperature field of a strip and a half strip connected at one end but with different thermophysical properties is determined for mixed conditions in the contact region.

1. Consider the problem of determining the steady temperature field for a half strip $x_1 \geq 0$ and a strip $|x_2| \leq h$, with the following conditions specified at their boundaries

$$T^{(1)}(x_1, \pm h) = f_1(x_1) \quad (x_1 \geqslant 0); \tag{1}$$

$$T^{(2)}(-b, x_2) = f_2(x_2); \tag{2}$$

$$\frac{\partial T^{(2)}}{\partial x_1}(0, x_2) = 0 \quad (|x_2| > h).$$

Here and below, the superscripts (1) and (2) denote characteristics of the halfstrip and strip, respectively; the function $f_2(x_2)$ is even.

The steady temperature field in each region is described by the following equations

$$\frac{d^2T^{(i)}}{\partial x_1^2} + k_i^2 \frac{\partial^2 T^{(i)}}{\partial x_2^2} = 0 \quad (i = 1, 2). \tag{3}$$

Suppose that, in some subregion S of the contact region [−h, h], there is ideal heat transfer, while the rest of the region is heat insulated. Then the matching conditions take the form

$$T^{(1)}(0, x_2) = T^{(2)}(0, x_2),$$
$$k_1^{(1)} \frac{\partial T^{(1)}}{\partial x_1}(0, x_2) = k_1^{(2)} \frac{\partial T^{(2)}}{\partial x_1}(0, x_2), \quad x_2 \in S, \tag{4}$$
$$\frac{\partial T^{(1)}}{\partial x_1}(0, x_2) = \frac{\partial T^{(2)}}{\partial x_1}(0, x_2) = 0, \quad x_2 \notin S.$$

To solve the problem, the unknown functions $\phi(x_2)$ and $\bar{\phi}(p)$ are introduced, according to the formulas

$$\varphi(x_2) = \left\{ \begin{array}{ll} k_1^{(1)} \dfrac{\partial T^{(1)}}{\partial x_1}(0, x_2), & x_2 \in S \\ 0, & x_2 \notin S \end{array} \right\} = \int_0^\infty \bar{\varphi}(p) \cos px_2 dp, \tag{5}$$

$$\bar{\varphi}(p) = \frac{2}{\pi} \int_0^\infty \varphi(x_2) \cos px_2 dx_2.$$

Then, applying the Fourier cos transformation [1] with respect to the coordinate x_1 to Eq. (3) (i = 1) and the boundary conditions in Eq. (1), and taking account of Eq. (5), it is found that

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$$\frac{\partial T^{(1)}}{\partial x_2}(0,\ x_2) = \frac{1}{\pi k_1 k_3}\int_S \varphi(y)\,dy\int_0^\infty \{\exp[-|x_2 - y|\,p/k_1] +$$

$$+\operatorname{sh}(px_2/k_1)\exp[-(h-y)\,p/k_1]/\operatorname{ch}(ph/k_1)\}\,dp + \frac{2}{\pi}V_1'(x_2),$$

where

$$\bar{f}_1(p) = \int_0^\infty f_1(x_1)\cos px_1 dx_1;\ \ k_3 = \sqrt{k_1^{(1)}k_2^{(1)}};$$

$$V_1(x_2) = \int_0^\infty \frac{\bar{f}_1(p)\operatorname{ch}(px_2/k_1)\,dp}{\operatorname{ch}(ph/k_1)}.$$

The solution of the boundary condition in Eqs. (2) and (3) (i = 2) and

$$\frac{\partial T^{(2)}}{\partial x_1}(0,\ x_2) = \begin{cases}\varphi(x_2)/k_1^{(2)}, & x_2 \in S,\\ 0, & x_2 \notin S\end{cases}$$

is obtained using a complex Fourier transformation [1] with respect to the coordinate x_2. The expression for the temperature derivative required in the subsequent analysis is

$$\frac{\partial T^{(2)}}{\partial x_2}(0,\ x_2) = -\frac{1}{\pi k_4}\int_S \varphi(y)\,dy\int_0^\infty \operatorname{th}(k_2 pb)\sin p(x_2 - y)\,dp \ -\frac{1}{\pi}V_2'(x_2),$$

where

$$V_2(x_2) = \int_0^\infty \frac{\bar{f}_2(p)\cos px_2 dp}{\operatorname{ch}(pk_2 b)};\ \ k_4 = \sqrt{k_1^{(2)}k_2^{(2)}};\ \bar{f}_2(p) = 2\int_0^\infty f_2(y)\cos py dy.$$

Satisfying the matching condition in Eq. (4), which is written in the form

$$\frac{\partial T^{(1)}}{\partial x_2}(0,\ x_2) = \frac{\partial T^{(2)}}{\partial x_2}(0,\ x_2),\ x_2 \in S,$$

where $T^{(1)}(0,\ v) = T^{(2)}(0,\ v)$ (v is some point of region S), the following singular integral equation is obtained

$$\int_S \left[\frac{1}{x_2 - y} + K(x_2,\ y)\right]\varphi(y)\,dy = F(x_2),\ x_2 \in S, \tag{6}$$

with the additional condition

$$\int_S K_0(y)\,\varphi(y)\,dy = F_0, \tag{7}$$

where

$$K(x_2,\ y) = \frac{1}{\beta k_1 k_3}\int_0^\infty \frac{\operatorname{sh}(px_2/k_1)\exp[-(h-y)\,p/k_1]\,dp}{\operatorname{ch}(ph/k_1)} - \frac{1}{\beta k_4}\int_0^\infty [1 - \operatorname{th}(k_2 pb)]\sin p(x_2 - y)\,dp;$$

$$K_0(y) = \frac{1}{k_4}\int_0^\infty \{\exp[-(h-y)\,p/k_1]\operatorname{ch}(pv/k_1) - \exp[-|v-y|\,p/k_1]\times$$

$$\times\operatorname{ch}(ph/k_1)\}/(p\operatorname{ch}(ph/k_1))\,dp - \frac{1}{k_4}\int_0^\infty \frac{1}{p}\operatorname{th}(k_2 pb)\cos p(v - y)\,dp;$$

$$F(x_2) = -[V_1'(x_2) + V_2'(x_2)]/\beta;$$

$$F_0 = \int_0^\infty \frac{\bar{f}_2(p)\cos pv\,dp}{\operatorname{ch}(k_2pb)} - 2\int_0^\infty \frac{\bar{f}_1(p)\operatorname{ch}(pv/k_1)\,dp}{\operatorname{ch}(ph/k_1)};$$

$$\beta = 1/k_3 + 1/k_4.$$

2. The function $K(x_2, y)$ may be written in the form

$$K(x_2, y) = \frac{1}{\beta k_3}\left[\frac{1}{2h - y - x_2} - \frac{1}{2h - y + x_2}\right] + K^*(x_2, y), \tag{8}$$

where $K^*(x_2, y) \in H$, H is a set of functions satisfying the Hölder condition in $[-h, h]$ [2]. This means that, if S includes corner points of the halfstrip, Eq. (6) includes not only mobile singularities but also immobile singularities as $x_2 \to \pm h$, $y \to h$. Suppose that the heat-insulated layer is at the section $[-a, a]$. This case will be called problem A. Then, taking account of the symmetry of $\phi(y)$, Eqs. (6) and (7) take the form

$$\int_a^h \left[\frac{2x_2}{x_2^2 - y^2} + M(x_2, y)\right]\varphi(y)\,dy = F(x_2), \; x_2 \in [a, h], \tag{9}$$

$$\int_a^h M_0(y)\varphi(y)\,dy = F_0, \tag{10}$$

where $M(x_2, y) = K(x_2, y) + K(x_2, -y)$; $M_0(y) = K_0(y) + K_0(-y)$.

Under the assumption that the unknown function has integrable singularities, $\phi(y)$ is sought in the form

$$\varphi(y) = \frac{\varphi^*(y)}{(y-a)^\gamma (h-y)^\alpha}, \; \varphi^*(y) \in H, \; 0 \leqslant \alpha, \; \gamma < 1.$$

Introducing the holomorphic function

$$\Phi(z) = \frac{1}{\pi}\int_{-h}^h \frac{\varphi(y)}{y - z}\,dy$$

the relations following from [2] are employed

$$\Phi(x_2) = \frac{\varphi^*(a)\operatorname{ctg}\pi\gamma}{(h-a)^\alpha (x_2-a)^\gamma} - \frac{\varphi^*(h)\operatorname{ctg}\pi\alpha}{(h-a)^\gamma (h-x_2)^\alpha} + \Phi_1^0(x_2), \; x_2 \to a, \; h;$$

$$\Phi(2h - x_2) = -\frac{\varphi^*(h)}{(h-a)^\gamma (h-x_2)^\alpha \sin\pi\alpha} + \Phi_2^0(x_2), \; x_2 \to h,$$

where $|\Phi_i^0(z)| \leqslant \dfrac{C_i}{(z-a)^{\gamma_0}(h-z)^{\alpha_0}}$; $\operatorname{Re}(\gamma_0) < \operatorname{Re}(\gamma)$; $\operatorname{Re}(\alpha_0) < \operatorname{Re}(\alpha)$; $C_i \; (i = 1, 2)$ are real constants.

Taking account of Eq. (8), the characteristic equations for determining the degrees of singularity α and γ are obtained from the condition that a nontrivial solution of Eq. (9) exists

$$\cos\pi\alpha + \frac{1}{\beta k_3} = 0, \quad \cos\pi\gamma = 0. \tag{11}$$

If the intervals $[-h, -a]$, $[a, h]$ are heat-insulated sections (problem B), then Eqs. (6) and (7) take the form

$$\int_{-a}^a \left[\frac{1}{x_2 - y} + K(x_2, y)\right]\varphi(y)\,dy = F(x_2), \; x_2 \in [-a, a],$$

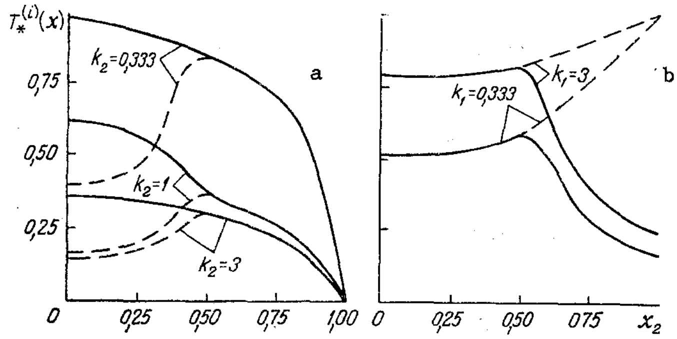

Fig. 1. Dimensionless temperature distribution in the contact zone with various values of the thermal conductivity of the strip, problem A (a), and the half strip, problem B (b).

$$\int_{-a}^{a} K_0(y)\varphi(y)\,dy = F_0.$$

In this case $\phi(y) = \phi^*(y)/(a^2 - y^2)^\alpha$, and $\alpha = 0.5$ if $a < h$ or is determined from Eq. (11) if $a = h$. In the latter case, there is no heat insulation in the contact zone.

A numerical method based on the Gauss–Jacobi quadrature formula of greatest algebraic accuracy is used to determine $\phi(y)$ and the temperature distribution in the contact zone [3]

$$\int_{-1}^{1} \frac{f(z, \tau)}{(1-\tau^2)^\alpha}\,d\tau = \sum_{k=1}^{N} A_k f(z, \tau_k), \tag{12}$$

a generalization of this formula to the case $f(z, \tau) = 1/(z - \tau)$ was given in [4]. Here τ_k are zeros of the Jacobi polynomial $P_N^{(-\alpha,-\gamma)}(\tau)$, A_k are constant coefficients.

The application of the method to Eqs. (9) and (10) will now be illustrated. Passing to the segment [−1, 1], choosing the zeros z_m (m = 1, 2, ..., N − 1) of the Jacobi polynomial $P_N^{(1-\alpha,1-\gamma)}(z)$ as the collocation points, and applying Eq. (12), the following system of linear algebraic equations is obtained

$$\sum_{k=1}^{N} A_k\left[\frac{T(z_m, \tau_k)}{z_m - \tau_k} + dM(c+dz_m,\ c+d\tau_k)\right]G(c+d\tau_k) = F(c+dz_m),$$

$$\sum_{k=1}^{N} A_k M_0(c+d\tau_k)\,G(c+d\tau_k) = F_0/d,$$

where $T(z, \tau) = 2(c + dz)/(d(z + \tau) + 2c)$; $G(c + d\tau_k) = (1 - \tau_k)^\alpha (1 + \tau_k)^\gamma d^{\alpha+\gamma}\phi(c + d\tau_k)$; $c = (h + a)/2$; $d = (h - a)2$.

The temperature values in the contact zones are determined by the formulas

$$T^{(1)}(0,\ x_2) = \frac{d}{\pi k_3}\sum_{k=1}^{N} A_k S_1(x_2,\ c+d\tau_k)\,G(c+d\tau_k) + \frac{2}{\pi}V_1(x_2),$$

$$T^{(2)}(0,\ x_2) = \frac{2d}{\pi k_4}\sum_{k=1}^{N} A_k S_2(x_2,\ c+d\tau_k)\,G(c+d\tau_k) + \frac{1}{\pi}V_2(x_2),$$

where

$$S_1(x_2,\ y) = \int_0^\infty \frac{1}{p}\left[\frac{\operatorname{ch}(px_2/k_1)}{\operatorname{ch}(ph/k_1)}\Omega(h,\ y,\ p) - \Omega(x_2,\ y,\ p)\right]dp;$$

$$S_2(x_2, y) = \int_0^\infty \frac{1}{p} \operatorname{th}(pbk_2) \cos py\,dp;$$

$$\Omega(x_2, y, p) = \exp[-|x_2 - y|\, p/k_1] + \exp[-|x_2 + y|p/k_1].$$

The results of calculating the temperature in the contact zone for problems A and B when h = b = 1, a = 0.5 are shown in Fig. 1. The continuous curves correspond to $T_*^{(2)}(x_*) = T^{(2)}(0, x_2)/T_0$ and the dashed curves to $T_*^{(1)}(x_2) = T^{(1)}(0, x_2)/T_0$. For Fig. 1a

$$f_1(x_1) = 0; \quad f_2(x_2) = \begin{cases} T_0, & |x_2| \leqslant 2, \\ 0, & |x_2| > 2, \end{cases} \quad k_1 = 1;$$

and for Fig. 1b

$$f_1(x_1) = \begin{cases} T_0, & x_1 \leqslant 2, \\ 0, & x_1 > 2, \end{cases} \quad f_2(x_2) = 0, \quad k_2 = 1.$$

These results illustrate the influence of the heat-insulating layer and the thermal conductivities on the temperature distribution in the contact zone.

NOTATION

x_1, x_2, axes of Cartesian coordinate system; $T^{(1)}$, $T^{(2)}$, temperature of half strip and strip; $k_1^{(i)}$, $k_2^{(i)}$ $(i = 1, 2)$, thermal conductivities in the directions x_1, x_2, respectively; 2h, b, width of strip and halfstrip, respectively; $\phi(x_2)$, heat flux in the contact region (basic unknown function); S, region of ideal thermal contact; $\{[-h, h] - S\}$, heat-insulated region; $k_i^2 = k_2^{(i)}/k_1^{(i)}$, $k_{i+2} = \sqrt{k_1^{(i)} k_2^{(i)}}$ $(i = 1, 2)$, $\beta = 1/k_3 + 1/k_4$, constants determined by the thermal conductivities; α, degree of singularity of the heat flux in the vicinity of the corners of the halfstrip; $P_N^{(\alpha,\beta)}(x)$, Jacobi polynomials; $f_1(x_1)$, $f_2(x_2)$, functions determining the temperature distribution at the boundaries of the regions.

LITERATURE CITED

1. N. S. Koshlyakov, É. B. Gliner, and M. M. Smirnov, Differential Equations of Mathematical Physics [in Russian], Moscow (1962).
2. N. I. Muskhelishvili, Singular Integral Equations [in Russian], Moscow (1968).
3. V. I. Krylov, Approximate Calculation of Integrals [in Russian], Moscow (1959).
4. A. A. Korneichuk, in: Numerical Methods of Solving Differential and Integral Equations and Quadrature Formulas [in Russian], Moscow (1964), pp. 64-74.